Normed and Banach spaces

In this chapter we introduce the basic setting of functional analysis, in the form of normed spaces and bounded linear operators. We are particularly interested in complete, i.e. Banach, spaces and the process of completion of a normed space to a Banach space.

1. Vector spaces

Note that for us the 'scalars' are either the real numbers or the complex numbers – usually the latter. To be neutral we denote by K either R or C, but of course consistently. Then our set V – the set of vectors with which we will deal, comes with two 'laws'. These are maps

$$+ : V \times V \longrightarrow V, \cdot : \mathrm{K} \times V \longrightarrow V. \tag{1.1}$$

which we denote not by $+(v,w)$ and $\cdot(s,v)$ but by $v+w$ and sv. Then we impose the *axioms of a vector space* – see Section (14) below! These are commutative group axioms for +, axioms for the action of K and the distributive law linking the two.

The basic examples:

- The field K which is either R or C is a vector space over itself.
- The vector spaces K^n consisting of ordered n-tuples of elements of K. Addition is by components and the action of K is by multiplication on all components. You should be reasonably familiar with these spaces and other finite dimensional vector spaces.
- Seriously non-trivial examples such as C([0,1]) the space of continuous functions on [0,1] (say with complex values).

In these and many other examples we will encounter below the 'component addition' corresponds to the addition of functions.

Lemma 1. *If X is a set then the spaces of all functions*

$$\mathrm{F}(X;\mathrm{R}) = \{u : X \longrightarrow \mathrm{R}\}, \mathrm{F}(X;\mathrm{C}) = \{u : X \longrightarrow \mathrm{C}\} \tag{1.2}$$

are vector spaces over R *and* C *respectively.*

The main point to make sure one understand is that, because we *do* know how to add and multiply in either R and C, we can add functions and multiply them by constants (we can multiply functions by each other but that is not part of the definition of a vector space so we ignore it for the moment since many of the spaces of functions we consider below are *not* multiplicative in this sense):-

$$(c_1f_1 + c_2f_2)(x) = c_1f_1(x) + c_2f_2(x) \text{ defines the function } c_1f_1 + c_2f_2 \tag{1.3}$$

$$\text{if } c_1, c_2 \in \mathrm{K} \text{ and } f_1, f_2 \in \mathrm{F}(X;\mathrm{K}).$$

Although you are probably most comfortable with finite-dimensional vector spaces it is the infinite-dimensional case that is most important here. The notion of dimension is based on the concept of the linear independence of a subset of a vector space. Thus a subset $E \subset V$ is said to be *linearly independent* if for any finite collection of elements $v_i \in E$, $i = 1,\dots,N$, and any collection of 'constants' $a_i \in \mathbb{K}$, $i = 1,\dots,N$ we have the following implication

$$\sum_{i=1}^{N} a_i v_i = 0 \Longrightarrow a_i = 0 \ \forall\ i. \tag{1.4}$$

That is, it is a set in which there are 'no non-trivial finite linear dependence relations between the elements'. A vector space is finite-dimensional if every linearly independent subset is finite. It follows in this case that there is a finite and maximal linearly independent subset – a basis – where maximal means that if any new element is added to the set E then it is no longer linearly independent. A basic result is that any two such 'bases' in a finite dimensional vector space have the same number of elements – an outline of the finite-dimensional theory can be found in Problem 1.

Definition 1. A vector space is infinite-dimensional if it is not finite dimensional, i.e. for any $N \in \mathbb{N}$ there exist N elements with no, non-trivial, linear dependence relation between them.

As is quite typical the idea of an infinite-dimensional space, which you may be quite keen to understand, appears just as the non-existence of something. That is, it is the 'residual' case, where there is no finite basis. This means that it is 'big'.

So, finite-dimensional vector spaces have finite bases, infinite-dimensional vector spaces do not. The notion of a basis in an infinite-dimensional vector spaces needs to be modified to be useful analytically. Convince yourself that the vector space in Lemma 1 is infinite dimensional if and only if X is infinite.

2. Normed spaces

In order to deal with infinite-dimensional vector spaces we need the control given by a metric (or more generally a non-metric topology, but we will not quite get that far). A norm on a vector space leads to a metric which is 'compatible' with the linear structure.

Definition 2. A norm on a vector space is a function, traditionally denoted

$$\|\cdot\| : V \longrightarrow [0,\infty), \tag{1.5}$$

with the following properties

(*Definiteness*)

$$v \in V,\ \|v\| = 0 \Longrightarrow v = 0. \tag{1.6}$$

(*Absolute homogeneity*) For any $\lambda \in \mathbb{K}$ and $v \in V$, (1.7) $$\|\lambda v\| = |\lambda|\|v\|.$$

(*Triangle Inequality*) The triangle inequality holds, in the sense that for any two elements $v, w \in V$

(1.8) $$\|v + w\| \le \|v\| + \|w\|.$$

Note that (1.7) implies that $\|0\| = 0$. Thus (1.6) means that $\|v\| = 0$ is equivalent to $v = 0$. This definition is based on the same properties holding for the standard norm(s), $|z|$, on $\mathbb{R}$ and $\mathbb{C}$. You should make sure you understand that

(1.9) $$\mathbb{R} \ni x \longrightarrow |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x \le 0 \end{cases} \in [0,\infty) \text{ is a norm as is}$$

$$\mathbb{C} \ni z = x + iy \longrightarrow |z| = (x^2 + y^2)^{\frac{1}{2}}.$$

Situations do arise in which we do not have (1.6):-

Definition 3. A function (1.5) which satisfes (1.7) and (1.8) but possibly not (1.6) is called a seminorm.

A metric, or distance function, on a set is a map

(1.10) $$d : X \times X \longrightarrow [0,\infty)$$

satisfying three standard conditions

(1.11) $$d(x,y) = 0 \Longleftrightarrow x = y,$$

(1.12) $$d(x,y) = d(y,x) \ \forall\ x,y \in X \text{ and}$$

(1.13) $$d(x,y) \le d(x,z) + d(z,y) \ \forall\ x,y,z \in X.$$

The point of course is Proposition 1. *If* $\|\cdot\|$ *is a norm on* V *then*

(1.14) $d(v,w) = \|v - w\|$ *is a distance on* V *turning it into a metric space.*

Proof. Clearly (1.11) corresponds to (1.6), (1.12) arises from the special case $\lambda = -1$ of (1.7) and (1.13) arises from (1.8).

We will not use any special notation for the metric, nor usually mention it explicitly – we just subsume all of metric space theory from now on. So $\|v - w\|$ is the distance between two points in a normed space.

Now, we need to talk about a few examples; there are more in Section 7. The most basic ones are the usual finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with their Euclidean norms

(1.15) $$|x| = \left(\sum_i |x_i|^2\right)^{\frac{1}{2}}$$

where it is at first confusing that we just use single bars for the norm, just as for $\mathbb{R}$ and $\mathbb{C}$, but you just need to get used to that.

There are other norms on $\mathbb{C}^n$ (I will mostly talk about the complex case, but the real case is essentially the same). The two most obvious ones are $|x|_\infty = \max|x_i|$, $x = (x_1,\ldots,x_n) \in \mathbb{C}^n$,

$$|x|_1 = \sum_i |x_i| \tag{1.16}$$

but as you will see (if you do the problems) there are also the norms

$$|x|_p = (\sum_i |x_i|^p)^{\frac{1}{p}},\ 1 \le p < \infty. \tag{1.17}$$

In fact, for $p = 1$, (1.17) reduces to the second norm in (1.16) and in a certain sense the case $p = \infty$ is consistent with the first norm there.

In lectures I usually do not discuss the notion of equivalence of norms straight away. However, two norms on the one vector space – which we can denote $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ are *equivalent* if there exist constants C_1 and C_2 such that

$$\|v\|_{(1)} \le C_1\|v\|_{(2)},\ \|v\|_{(2)} \le C_2\|v\|_{(1)}\ \forall\ v \in V. \tag{1.18}$$

The equivalence of the norms implies that the metrics define the same open sets – the topologies induced are the same. You might like to check that the reverse is also true, if two norms induced the same topologies (just meaning the same collection of open sets) through their associated metrics, then they are equivalent in the sense of (1.18) (there are more efficient ways of doing this if you wait a little).

One important class of normed spaces consists of the spaces of bounded continuous functions on a metric space X :

$$\mathcal{C}_\infty(X) = \mathcal{C}_\infty(X;\mathbb{C}) = \{u : X \longrightarrow \mathbb{C},\ \text{continuous and bounded}\}. \tag{1.19}$$

That this is a linear space follows from the (obvious) result that a linear combination of bounded functions is bounded and the (less obvious) result that a linear combination of continuous functions is continuous; this we know. The norm is the best bound

$$\|u\|_\infty = \sup_{x\in X} |u(x)|. \tag{1.20}$$

That this *is* a norm is straightforward to check. Absolute homogeneity is clear, $\|\lambda u\|_\infty = |\lambda|\|u\|_\infty$ and $\|u\|_\infty = 0$ means that $u(x) = 0$ for all $x \in X$ which is exactly what it means for a function to vanish. The triangle inequality 'is inherited from $\mathbb{C}$' since for any two functions and any point,

$$|(u + v)(x)| \le |u(x)| + |v(x)| \le \|u\|_\infty + \|v\|_\infty \tag{1.21}$$

by the definition of the norms, and taking the supremum of the left gives

$$\|u + v\|_\infty \le \|u\|_\infty + \|v\|_\infty.$$

Of course the norm (1.20) is defined even for bounded, not necessarily continuous functions on X. Note that convergence of a sequence $u_n \in \mathcal{C}_\infty(X)$ (remember this means with respect to the distance induced by the norm) is precisely *uniform convergence*

(1.22) $$\|u_n - v\|_\infty \to 0 \Longleftrightarrow u_n(x) \to v(x) \text{ uniformly on } X.$$

Other examples of infinite-dimensional normed spaces are the spaces l^p, $1 \le p \le \infty$ discussed in the problems below. Of these l^2 is the most important for us. It is in fact one form of Hilbert space, with which we are primarily concerned:-

(1.23) $$l^2 = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_j |a(j)|^2 < \infty\}.$$

It is not immediately obvious that this is a linear space, nor that

(1.24) $$\|a\|_2 = \left(\sum_j |a(j)|^2\right)^{\frac{1}{2}}$$

is a norm. It is. From now on we will generally use sequential notation and think of a map from $\mathbb{N}$ to $\mathbb{C}$ as a sequence, so setting $a(j) = a_j$. Thus the 'Hilbert space' l^2 consists of the square summable sequences.

3. Banach spaces

Definition 4. A normed space which is complete with respect to the induced metric is a *Banach* space.

Lemma 2. *The space* $\mathcal{C}_\infty(X)$*, defined in* (1.19) *for any metric space X, is a Banach space.*

Proof. This is a standard result from metric space theory – basically that the uniform limit of a sequence of (bounded) continuous functions on a metric space is continuous. However, it is worth recalling how one proves completeness at least in outline. Suppose u_n is a Cauchy sequence in $\mathcal{C}_\infty(X)$. This means that given $\delta > 0$ there exists N such that

(1.25) $$n,m > N \Longrightarrow \|u_n - u_m\|_\infty = \sup_X |u_n(x) - u_m(x)| < \delta.$$

Fixing $x \in X$ this implies that the sequence $u_n(x)$ is Cauchy in $\mathbb{C}$. We know that this space is complete, so each sequence $u_n(x)$ must converge (we say the sequence of functions converges pointwise). Since the limit of $u_n(x)$ can only depend on x, we define $u(x) = \lim_n u_n(x)$ in $\mathbb{C}$ for each $x \in X$ and so define a function $u : X \longrightarrow \mathbb{C}$. Now, we need to show that this is bounded and continuous and is the limit of u_n with respect to the norm. Any Cauchy sequence is bounded in norm – take $\delta = 1$ in (1.25) and it follows from the triangle inequality that

(1.26) $$\|u_m\|_\infty \le \|u_{N+1}\|_\infty + 1,\ m > N$$

and the finite set $\|u_n\|_\infty$ for $n \le N$ is certainly bounded. Thus $\|u_n\|_\infty \le C$, but this means $|u_n(x)| \le C$ for all $x \in X$ and hence $|u(x)| \le C$ by properties of convergence in $\mathbb{C}$ and thus $\|u\|_\infty \le C$.

The uniform convergence of u_n to u now follows from (1.25) since we may pass to the limit in the inequality to find

(1.27) $$n > N \Longrightarrow |u_n(x) - u(x)| = \lim_{m\to\infty} |u_n(x) - u_m(x)| \le \delta$$

$$\Longrightarrow \|u_n - u\|_{\infty} \le \delta.$$

The continuity of u at $x \in X$ follows from the triangle inequality in the form

$$|u(y) - u(x)| \le |u(y) - u_n(y)| + |u_n(y) - u_n(x)| + |u_n(x) - u_n(x)|$$
$$\le 2\|u - u_n\|_\infty + |u_n(x) - u_n(y)|.$$

Give $\delta > 0$ the first term on the far right can be make less than $\delta/2$ by choosing n large using (1.27) and then the second term can be made less than $\delta/2$ by choosing $d(x,y)$ small enough.

There is a space of sequences which is really an example of this Lemma. Consider the space c_0 consisting of all the sequences $\{a_j\}$ (valued in $\mathbb{C}$) such that $\lim_{j\to\infty} a_j = 0$. As remarked above, sequences are just functions $\mathbb{N} \longrightarrow \mathbb{C}$. If we make $\{a_j\}$ into a function $\alpha : D = \{1,1/2,1/3,...\} \longrightarrow \mathbb{C}$ by setting $\alpha(1/j) = a_j$ then we get a function on the metric space D. Add 0 to D to get

$\overline{D} = D \cup \{0\} \subset [0,1] \subset \mathbb{R}$; clearly 0 is a limit point of D and $\overline{D}$ is, as the notation dangerously indicates, the closure of D in $\mathbb{R}$. Now, you will easily check (it is really the definition) that $\alpha : D \longrightarrow \mathbb{C}$ corresponding to a sequence, extends to a

continuous function on $\overline{D}$ vanishing at 0 if and only if $\lim_{j\to\infty} a_j = 0$, which is to say, $\{a_j\} \in c_0$. Thus it follows, with a little thought which you should give it, that c_0 is a Banach space with the norm

$$\|a\|_\infty = \sup_j \|a_j\|. \tag{1.28}$$

3. BANACH SPACES

What is an example of a non-complete normed space, a normed space which is *not* a Banach space? These are legion of course. The simplest way to get one is to 'put the wrong norm' on a space, one which does not correspond to the definition. Consider for instance the linear space T of sequences $\mathbb{N} \longrightarrow \mathbb{C}$ which 'terminate', i.e. each element $\{a_j\} \in T$ has $a_j = 0$ for $j > J$, where of course the J may depend on the particular sequence. Then $T \subset c_0$, the norm on c_0 defines a norm on T but it cannot be complete, since the closure of T is easily seen to be all of c_0 – so there are Cauchy sequences in T without limit in T . Make sure you are not lost here – you need to get used to the fact that we often need to discuss the 'convergence of sequences of convergent sequences' as here.

One result we will exploit below, and I give it now just as preparation, concerns *absolutely summable series*. Recall that a series is just a sequence where we 'think' about adding the terms. Thus if v_n is a sequence in some vector space V then there

is the corresponding sequence of partial sums $w_N = \sum_{i=1}^{N} v_i$. I will say that $\{v_n\}$ is a series if I am thinking about summing it.

So a sequence $\{v_n\}$ with partial sums $\{w_N\}$ is said to be *absolutely summable* if

(1.29), i.e. $\sum_n \|v_n\|_V < \infty$ $\quad \sum_{N>1} \|w_N - w_{N-1}\|_V < \infty.$

Proposition 2. *The sequence of partial sums of any absolutely summable series in a normed space is Cauchy and a normed space is complete if and only if every absolutely summable series in it converges, meaning that the sequence of partial sums converges.*

Proof. The sequence of partial sums is

$$w_n = \sum_{j=1}^{n} v_j. \tag{1.30}$$

Thus, if $m \geq n$ then

$$w_m - w_n = \sum_{j=n+1}^{m} v_j. \tag{1.31}$$

It follows from the triangle inequality that

$$\|w_n - w_m\|_V \leq \sum_{j=n+1}^{m} \|v_j\|_V. \tag{1.32}$$

So if the series is absolutely summable then

$$\sum_{j=1}^{\infty} \|v_j\|_V < \infty \text{ and } \lim_{n\to\infty} \sum_{j=n+1}^{\infty} \|v_j\|_V = 0.$$

Thus $\{w_n\}$ is Cauchy if $\{v_j\}$ is absolutely summable. Hence if V is complete then every absolutely summable series is summable, i.e. the sequence of partial sums converges.

Conversely, suppose that every absolutely summable series converges in this sense. Then we need to show that every Cauchy sequence in V converges. Let u_n be a Cauchy sequence. It suffices to show that this has a subsequence which converges, since a Cauchy sequence with a convergent subsequence is convergent.

To do so we just proceed inductively. Using the Cauchy condition we can for every k find an integer N_k such that

$$n, m > N_k \Longrightarrow \|u_n - u_m\| < 2^{-k}. \tag{1.33}$$

Now choose an increasing sequence n_k where $n_k > N_k$ and $n_k > n_{k-1}$ to make it increasing. It follows that

$$\|u_{n_k} - u_{n_{k-1}}\| \leq 2^{-k+1}. \tag{1.34}$$

Denoting this subsequence as $u'_k = u_{n_k}$ it follows from (1.34) and the triangle inequality that

$$\sum_{n=1}^{\infty} \|u'_n - u'_{n-1}\| \leq 4 \tag{1.35}$$

so the sequence $v_1 = u'_1,\ v_k = u'_k - u'_{k-1},\ k > 1$, is absolutely summable. Its sequence of partial sums is $w_j = u'_j$ so the assumption is that this converges, hence the original Cauchy sequence converges and V is complete.

Notice the idea here, of 'speeding up the convergence' of the Cauchy sequence by dropping a lot of terms. We will use this idea of absolutely summable series heavily in the discussion of Lebesgue integration.

4. Operators and functionals

The vector spaces we are most interested in are, as already remarked, spaces of functions (or something a little more general). The elements of these are the objects of primary interest but they are related by linear maps. A map between two vector spaces (over the same field, for us either R or C) is linear if it takes linear combinations to linear combinations:-

(1.36) $T : V \longrightarrow W,\ T(a_1v_1+a_2v_2) = a_1T(v_1)+a_2T(v_2),\ \forall\ v_1, v_2 \in V,\ a_1,a_2 \in \mathbb{K}.$

The sort of examples we have in mind are differential, or more especially, integral operators. For instance if $u \in C([0,1])$ then its indefinite Riemann integral

(1.37) $$(Tu)(x) = \int_0^x u(s)ds$$

is continuous in $x \in [0,1]$ and so this defines a map

(1.38) $$T : C([0,1]) \longrightarrow C([0,1]).$$

This is a linear map, with linearity being one of the standard properties of the Riemann integral.

In the finite-dimensional case linearity is enough to allow maps to be studied. However in the case of infinite-dimensional normed spaces we need to impose continuity. Of course it makes perfectly good sense to say, demand or conclude, that a map as in (1.36) is continuous if V and W are normed spaces since they are then

metric spaces. Recall that for metric spaces there are several different equivalent conditions that ensure a map, $T : V \longrightarrow W$, is continuous:

(1.39) $$v_n \to v \text{ in } V \Longrightarrow Tv_n \to Tv \text{ in } W$$

(1.40) $$O \subset W \text{ open} \Longrightarrow T^{-1}(O) \subset V \text{ open}$$

(1.41) $$C \subset W \text{ closed} \Longrightarrow T^{-1}(C) \subset V \text{ closed}.$$

For a linear map between normed spaces there is a direct characterization of continuity in terms of the norm.

Proposition 3. *A linear map* (1.36) *between normed spaces is continuous if and only if it is* bounded *in the sense that there exists a constant C such that*

(1.42) $$\|Tv\|_W \le C\|v\|_V \; \forall \; v \in V.$$

Of course bounded for a function on a metric space already has a meaning and this is not it! The usual sense would be $\|Tv\| \le C$ but this would imply $\|T(av)\| = |a|\|Tv\| \le C$ so $Tv = 0$. Hence it is not so dangerous to use the term 'bounded' for (1.42) – it is really 'relatively bounded', i.e. takes bounded sets into bounded sets. From now on, bounded for a linear map means (1.42).

Proof. If (1.42) holds then if $v_n \to v$ in V it follows that $\|Tv - Tv_n\| = \|T(v - v_n)\| \le C\|v - v_n\| \to 0$ as $n \to \infty$ so $Tv_n \to Tv$ and continuity follows.

For the reverse implication we use the second characterization of continuity above. Denote the ball around $v \in V$ of radius $\epsilon > 0$ by

$$B_V(v,\epsilon) = \{w \in V; \|v - w\| < \epsilon\}.$$

Thus if T is continuous then the inverse image of the the unit ball around the origin, $T^{-1}(B_W(0,1)) = \{v \in V; \|Tv\|_W < 1\}$, contains the origin in V and so, being open, must contain some $B_V(0,\epsilon)$. This means that

(1.43) $$T(B_V(0,\epsilon)) \subset B_W(0,1) \text{ so } \|v\|_V < \epsilon \Longrightarrow \|Tv\|_W \le 1.$$

Now proceed by scaling. If $0 \ne v \in V$ then $\|v'\| < \epsilon$ where $v' = \epsilon v/2\|v\|$. So (1.43) shows that $\|Tv'\| \le 1$ but this implies (1.42) with $C = 2/\epsilon$ – it is trivially true if $v = 0$.

As a general rule we drop the distinguishing subscript for norms, since which norm we are using can be determined by what it is being applied to.

So, if $T : V \longrightarrow W$ is continous and linear between normed spaces, or from now on 'bounded', then

(1.44) $$\|T\| = \sup_{\|v\|=1} \|Tv\| < \infty.$$

Lemma 3. *The bounded linear maps between normed spaces V and W form a linear space* $\mathcal{B}(V,W)$ *on which* $\|T\|$ *defined by* (1.44) *or equivalently*

(1.45) $\|T\| = \inf\{C;$ (1.42) *holds*$\}$ *is a norm.*

Proof. First check that (1.44) is equivalent to (1.45). Define $\|T\|$ by (1.44). Then for any $v \in V$, $v \ne 0$,

(1.46) $$\|T\| \ge \|T(\frac{v}{\|v\|})\| = \frac{\|Tv\|}{\|v\|} \Longrightarrow \|Tv\| \le \|T\|\|v\|$$

since as always this is trivially true for $v = 0$. Thus $C = \|T\|$ is a constant for which (1.42) holds.

Conversely, from the definition of $\|T\|$, if $\epsilon > 0$ then there exists $v \in V$ with $\|v\| = 1$ such that $\|T\| - \epsilon < \|Tv\| \le C$ for any C for which (1.42) holds. Since $\epsilon > 0$ is arbitrary, $\|T\| \le C$ and hence $\|T\|$ is given by (1.45).

From the definition of $\|T\|$, $\|T\| = 0$ implies $Tv = 0$ for all $v \in V$ and for $\lambda \ne 0$,

(1.47) $$\|\lambda T\| = \sup_{\|v\|=1} \|\lambda Tv\| = |\lambda|\|T\|$$

and this is also obvious for $\lambda = 0$. This only leaves the triangle inequality to check and for any $T, S \in \mathcal{B}(V,W)$, and $v \in V$ with $\|v\| = 1$

(1.48) $\quad \|(T+S)v\|_W = \|Tv + Sv\|_W \le \|Tv\|_W + \|Sv\|_W \le \|T\| + \|S\|$

so taking the supremum, $\|T+S\| \le \|T\| + \|S\|$.

Thus we see the very satisfying fact that the space of bounded linear maps between two normed spaces is itself a normed space, with the norm being the best constant in the estimate (1.42). Make sure you absorb this! Such bounded linear maps between normed spaces are often called ‘operators’ because we are thinking of the normed spaces as being like function spaces.

You might like to check boundedness for the example of a linear operator in (1.38), namely that in terms of the supremum norm on $C([0,1])$, $\|T\| \le 1$.

One particularly important case is when $W = \mathbb{K}$ is the field, for us usually $\mathbb{C}$. Then a simpler notation is handy and one sets $V^0 = B(V,\mathbb{C})$ – this is called the *dual space* of V (also sometimes denoted V^*.)

Proposition 4. *If W is a Banach space then $B(V,W)$, with the norm (1.44), is a Banach space.*

Proof. We simply need to show that if W is a Banach space then every Cauchy sequence in $B(V,W)$ is convergent. The first thing to do is to find the limit. To say that $T_n \in B(V,W)$ is Cauchy, is just to say that given > 0 there exists N such that $n, m > N$ implies $\|T_n - T_m\| < \epsilon$. By the definition of the norm, if $v \in V$
then $\|T_n v - T_m v\|_W \le \|T_n - T_m\|\|v\|_V$ so $T_n v$ is Cauchy in W for each $v \in V$. By assumption, W is complete, so

(1.49) $$T_n v \longrightarrow w \text{ in } W.$$

However, the limit can only depend on v so we can define a map $T : V \longrightarrow W$ by
$Tv = w = \lim_{n\to\infty} T_n v$ as in (1.49).

This map defined from the limits is linear, since $T_n(\lambda v) = \lambda T_n v \longrightarrow \lambda Tv$ and $T_n(v_1+v_2) = T_n v_1 + T_n v_2 \longrightarrow Tv_2 + Tv_2 = T(v_1+v_2)$. Moreover, $|\|T_n\| - \|T_m\|| \le \|T_n - T_m\|$ so $\|T_n\|$ is Cauchy in $[0,\infty)$ and hence converges, with limit S, and

(1.50) $$\|Tv\| = \lim_{n\to\infty} \|T_n v\| \le S\|v\|$$

so $\|T\| \le S$ shows that T is bounded.

Returning to the Cauchy condition above and passing to the limit in $\|T_n v - T_m v\| \le \epsilon\|v\|$ as $m \to \infty$ shows that $\|T_n - T\| \le \epsilon$ if $n > M$ and hence $T_n \to T$ in
$B(V,W)$ which is therefore complete.

Note that this proof is structurally the same as that of Lemma 2. One simple consequence of this is:-

Corollary 1. *The dual space of a normed space is always a Banach space.*

However we should be a little suspicious here since we have not shown that the dual space V^0 is non-trivial, meaning we have not eliminated the possibility that $V^0 = \{0\}$ even when $V \neq \{0\}$. The Hahn-Banach Theorem, discussed below, takes care of this.

One game you can play is ‘what is the dual of that space’. Of course, the dual is the dual, but you may well be able to identify the dual space of V with some other Banach space by finding a linear bijection between V^0 and the other space, W, which identifies the norms as well. We will play this game a bit later.

5. Subspaces and quotients

The notion of a linear subspace of a vector space is natural enough, and you are likely quite familiar with it. Namely $W \subset V$ where V is a vector space is a (linear) subspace if any linear combinations $\lambda_1 w_1 + \lambda_2 w_2 \in W$ if $\lambda_1, \lambda_2 \in \mathbb{K}$ and $w_1, w_2 \in W$. Thus W ‘inherits’ its linear structure from V. Since we also have a topology from the metric we will be especially interested in closed subspaces. Check that you understand the (elementary) proof of

Lemma 4. *A subspace of a Banach space is a Banach space in terms of the restriction of the norm if and only if it is closed.*

There is a second very important way to construct new linear spaces from old. Namely we want to make a linear space out of ‘the rest’ of V, given that W is a linear subspace. In finite dimensions one way to do this is to give V an inner product and then take the subspace orthogonal to W. One problem with this is that the result depends, although not in an essential way, on the inner product. Instead we adopt the usual ‘myopia’ approach and take an equivalence relation on V which identifies points which differ by an element of W. The equivalence classes are then ‘planes parallel to W’. I am going through this construction quickly here under the assumption that it is familiar to most of you, if not you should think about it carefully since we need to do it several times later.

So, if $W \subset V$ is a linear subspace of V we define a relation on V – remember this is just a subset of $V \times V$ with certain properties – by

$$v \sim_W v^0 \Longleftrightarrow v - v^0 \in W \Longleftrightarrow \exists\ w \in W \text{ s.t. } v = v^0 + w. \tag{1.51}$$

This satisfies the three conditions for an equivalence relation:

(1) $v \sim_W v$

(2) $v \sim_W v^0 \Longleftrightarrow v^0 \sim_W v$

(3) $v \sim_W v_0,\ v_0 \sim_W w_{00} \Longrightarrow v \sim_W v_{00}$

which means that we can regard it as a ‘coarser notion of equality.’

Then V/W is the set of equivalence classes with respect to $\sim_W$. You can think of the elements of V/W as being of the form $v + W$ – a particular element of V plus an arbitrary element of W. Then of course $v^0 \in v+W$ if and only if $v^0 - v \in W$ meaning $v \sim_W v^0$.

The crucial point here is that

$$V/W \text{ is a vector space.} \tag{1.52}$$

You should check the details – see Problem 1. Note that the ‘is’ in (1.52) should really be expanded to ‘is in a natural way’ since as usual the linear structure is inherited from V:

$$\lambda(v + W) = \lambda v + W,\ (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W. \tag{1.53}$$

The subspace W appears as the origin in V/W.

Now, two cases of this are of special interest to us.

Proposition 5. *If* k · k *is a seminorm on* V *then*

$$E = \{v \in V ; \mathrm{k}v\mathrm{k} = 0\} \subset V \tag{1.54}$$

is a linear subspace and

$$\mathrm{k}v + E\mathrm{k}_{V/E} = \mathrm{k}v\mathrm{k} \tag{1.55}$$

defines a norm on V/E.

Proof. That E is linear follows from the properties of a seminorm, since $\mathrm{k}\lambda v\mathrm{k} = |\lambda|\mathrm{k}v\mathrm{k}$ shows that $\lambda v \in E$ if $v \in E$ and $\lambda \in \mathrm{K}$. Similarly the triangle inequality shows that $v_1 + v_2 \in E$ if $v_1, v_2 \in E$.

To check that (1.55) defines a norm, first we need to check that it makes sense as a function $\mathrm{k} \cdot \mathrm{k}_{V/E} \longrightarrow [0,\infty)$. This amounts to the statement that $\mathrm{k}v^0\mathrm{k}$ is the same for all elements $v^0 = v + e \in v + E$ for a fixed v. This however follows from the triangle inequality applied twice:

$$\mathrm{k}v^0\mathrm{k} \le \mathrm{k}v\mathrm{k} + \mathrm{k}e\mathrm{k} = \mathrm{k}v\mathrm{k} \le \mathrm{k}v^0\mathrm{k} + \mathrm{k} - e\mathrm{k} = \mathrm{k}v^0\mathrm{k}. \tag{1.56}$$

Now, I leave you the exercise of checking that $\mathrm{k}\cdot\mathrm{k}_{V/E}$ is a norm, see Problem 1.

The second application is more serious, but in fact we will not use it for some time so I usually do not do this in lectures at this stage.

Proposition 6. *If* $W \subset V$ *is a closed subspace of a normed space then*

$$\|v + W\|_{V/W} = \inf_{w \in W} \|v + w\|_V \tag{1.57}$$

defines a norm on V/W; if V is a Banach space then so is V/W.

For the proof see Problems 1 and 1.

6. Completion

A normed space not being complete, not being a Banach space, is considered to be a defect which we might, indeed will, wish to rectify.

Let V be a normed space with norm $\mathrm{k} \cdot \mathrm{k}_V$. A *completion* of V is a Banach space B with the following properties:-

(1) There is an injective (i.e. 1-1) linear map $I : V \longrightarrow B$
(2) The norms satisfy

$$\mathrm{k}I(v)\mathrm{k}_B = \mathrm{k}v\mathrm{k}_V \;\forall\; v \in V. \tag{1.58}$$

(3) The range $I(V) \subset B$ is dense in B.

Notice that if V is itself a Banach space then we can take $B = V$ with I the identity map.

So, the main result is:

Theorem 1. *Each normed space has a completion.*

There are several ways to prove this, we will come across a more sophisticated one (using the Hahn-Banach Theorem) later. In the meantime I will give two proofs. In the first the fact that any metric space has a completion in a similar sense is recalled and then it is shown that the linear structure extends to the completion. A second, 'hands-on', proof is also outlined with the idea of motivating the construction of the Lebesgue integral – which is in our near future.

Proof 1. One of the neater proofs that any metric space has a completion is to use Lemma 2. Pick a point in the metric space of interest, $p \in M$, and then define a map

$$M\ 3\ q\ 7\!-\!\!\to f_q \in \mathrm{C}_\infty(M),\ f_q(x) = d(x,q) - d(x,p)\ \forall\ x \in M. \tag{1.59}$$

That $f_q \in \mathrm{C}_\infty(M)$ is straightforward to check. It is bounded (because of the second term) by the reverse triangle inequality

$$|f_q(x)| = |d(x,q) - d(x,p)| \le d(p,q)$$

and is continuous, as the difference of two continuous functions. Moreover the distance between two functions in the image is

$$\sup_{x\in M} |f_q(x) - f_{q^0}(x)| = \sup_{x\in M} |d(x,q) - d(x,q^0)| = d(q,q^0) \tag{1.60}$$

using the reverse triangle inequality (and evaluating at $x = q$). Thus the map (1.59) is well-defined, injective and even distance-preserving. Since $\mathrm{C}_\infty{}^0(M)$ is complete, the closure of the image of (1.59) is a complete metric space, X, in which M can be identified as a dense subset.

Now, in case that $M = V$ is a normed space this all goes through. The disconcerting thing is that the map $q \longrightarrow f_q$ is *not* linear. Nevertheless, we can give X a linear structure so that it becomes a Banach space in which V is a dense linear subspace. Namely for any two elements $f_i \in X$, $i = 1,2$, define

$$\lambda_1 f_1 + \lambda_2 f_2 = \lim f_{\lambda_1 p_n + \lambda_2 q_n}\ {}_{n\to\infty} \tag{1.61}$$

where p_n and q_n are sequences in V such that $f_{p_n} \to f_1$ and $f_{q_n} \to f_2$. Such sequences exist by the construction of X and the result does not depend on the choice of sequence – since if $p^0{}_n$ is another choice in place of p_n then $f_{p^0{}_n} - f_{p_n} \to 0$ in X (and similarly for q_n). So the element of the left in (1.61) is well-defined. All of the properties of a linear space and normed space now follow by continuity from $V \subset X$ and it also follows that X is a Banach space (since a closed subset of a complete space is complete). Unfortunately there are quite a few annoying details to check!

'Proof 2' (the last bit is left to you). Let V be a normed space. First we introduce the rather large space

$$\widetilde{V} = \left\{ \{u_k\}_{k=1}^{\infty};\ u_k \in V \text{ and } \sum_{k=1}^{\infty} \|u_k\| < \infty \right\} \tag{1.62}$$

the elements of which, if you recall, are said to be absolutely summable. Notice that the elements of Ve are *sequences*, valued in V so two sequences are equal, are the same, only when each entry in one is equal to the corresponding entry in the other – no shifting around or anything is permitted as far as equality is concerned. We think of these as series (remember this means nothing except changing the name, a series is a sequence and a sequence is a series), the only difference is that we 'think' of taking the limit of a sequence but we 'think' of summing the elements of a series, whether we can do so or not being a different matter.

Now, each element of $\tilde{V}$ is a Cauchy sequence – meaning the corresponding sequence of partial sums $v_N = \sum_{k=1}^{N} u_k$ is Cauchy if $\{u_k\}$ is absolutely summable. As noted earlier, this is simply because if $M \geq N$ then

(1.63) $$\|v_M - v_N\| = \|\sum_{j=N+1}^{M} u_j\| \leq \sum_{j=N+1}^{M} \|u_j\| \leq \sum_{j\geq N+1} \|u_j\|$$

gets small with N by the assumption that $\sum_j \|u_j\| < \infty$.

Moreover, $\tilde{V}$ is a linear space, where we add sequences, and multiply by constants, by doing the operations on each component:-

(1.64) $$t_1\{u_k\} + t_2\{u'_k\} = \{t_1 u_k + t_2 u'_k\}.$$

This always gives an absolutely summable series by the triangle inequality:

(1.65) $$\sum_k \|t_1 u_k + t_2 u'_k\| \leq |t_1| \sum_k \|u_k\| + |t_2| \sum_k \|u'_k\|.$$

Within $\tilde{V}$ consider the linear subspace

(1.66) $$S = \left\{ \{u_k\}; \sum_k \|u_k\| < \infty,\ \sum_k u_k = 0 \right\}$$

of those which sum to 0. As discussed in Section 5 above, we can form the quotient

(1.67) $$B = \tilde{V}/S$$

the elements of which are the 'cosets' of the form $\{u_k\} + S \subset \tilde{V}$ where $\{u_k\} \in \tilde{V}$. This is our completion, we proceed to check the following properties of this B.

(1) A norm on B (via a seminorm on $\tilde{V}$) is defined by

(1.68) $$\|b\|_B = \lim_{n\to\infty} \|\sum_{k=1}^{n} u_k\|,\ b = \{u_k\} + S \in B.$$

(2) The original space V is imbedded in B by

(1.69) $$V \ni v \longmapsto I(v) = \{u_k\} + S,\ u_1 = v,\ u_k = 0\ \forall\ k > 1$$

and the norm satisfies (1.58).

(3) $I(V) \subset B$ is dense.

(4) B is a Banach space with the norm (1.68).

So, first that (1.68) is a norm. The limit on the right does exist since the limit of the norm of a Cauchy sequence always exists – namely the sequence of norms is itself Cauchy but now in $\mathbb{R}$. Moreover, adding an element of S to $\{u_k\}$ does not change the norm of the sequence of partial sums, since the additional term tends to zero in norm. Thus $\|b\|_B$ is well-defined for each element $b \in B$ and $\|b\|_B = 0$ means exactly that the sequence $\{u_k\}$ used to define it tends to 0 in norm, hence is in S hence $b = 0$ in B. The other two properties of norm are reasonably clear, since

if $b, b' \in B$ are represented by $\{u_k\}$, $\{u'_k\}$ in $\widetilde{V}$ then tb and $b + b'$ are represented by $\{tu_k\}$ and $\{u_k + u'_k\}$ and

(1.70)

$$\lim_{n\to\infty} \|\sum_{k=1}^{n} tu_k\| = |t| \lim_{n\to\infty} \|\sum_{k=1}^{n} u_k\|, \Longrightarrow \|tb\| = |t|\|b\|$$

$$\lim_{n\to\infty} \|\sum_{k=1}^{n}(u_k + u'_k)\| = A \Longrightarrow$$

$$\text{for } \epsilon > 0 \ \exists\ N \text{ s.t. } \forall\ n \geq N,\ A - \epsilon \leq \|\sum_{k=1}^{n}(u_k + u'_k)\| \Longrightarrow$$

$$A - \epsilon \leq \|\sum_{k=1}^{n} u_k\| + \|\sum_{k=1}^{n} u'_k)\|\ \forall\ n \geq N \Longrightarrow A - \epsilon \leq \|b\|_B + \|b'\|_B\ \forall\ \epsilon > 0 \Longrightarrow$$

$$\|b + b'\|_B \leq \|b\|_B + \|b'\|_B.$$

Now the norm of the element $I(v) = v, 0, 0, \cdots$, is the limit of the norms of the sequence of partial sums and hence is $\|v\|_V$ so $\|I(v)\|_B = \|v\|_V$ and $I(v) = 0$ therefore implies $v = 0$ and hence I is also injective.

We need to check that B is complete, and also that $I(V)$ is dense. Here is an extended discussion of the difficulty – of course maybe you can see it directly yourself (or have a better scheme). Note that I suggest that you to write out your own version of it carefully in Problem 1.

Okay, what does it mean for B to be a Banach space, as discussed above it means that every absolutely summable series in B is convergent. Such a series $\{b_n\}$ is given by $b_n = \{u_k^{(n)}\} + S$ where $\{u_k^{(n)}\} \in \widetilde{V}$ and the summability condition is that

$$\infty > \sum_n \|b_n\|_B = \sum_n \lim_{N\to\infty} \|\sum_{k=1}^{N} u_k^{(n)}\|_V. \tag{1.71}$$

So, we want to show that $\sum_n b_n = b$ converges, and to do so we need to find the limit b. It is supposed to be given by an absolutely summable series. The 'problem' is that this series should look like $\sum_n \sum_k u_k^{(n)}$ in some sense – because it is supposed to represent the sum of the b_n's. Now, it would be very nice if we had the estimate

$$\sum_n \sum_k \|u_k^{(n)}\|_V < \infty \tag{1.72}$$

since this should allow us to break up the double sum in some nice way so as to get an absolutely summable series out of the whole thing. The trouble is that (1.72) need not hold. We know that *each* of the sums over k – for given n – converges, but not the sum of the sums. All we know here is that the sum of the 'limits of the norms' in (1.71) converges.

So, that is the problem! One way to see the solution is to note that we do not have to choose the original $\{u_k^{(n)}\}$ to 'represent' b_n – we can add to it any element of S. One idea is to rearrange the $u_k^{(n)}$ – I am thinking here of fixed n – so that it 'converges even faster.' I will not go through this in full detail but rather do it later when we need the argument for the completeness of the space of Lebesgue integrable functions. Given $\epsilon > 0$ we can choose p_1 so that for all $p \geq p_1$,

$$\Big|\|\sum_{k\leq p} u_k^{(n)}\|_V - \|b_n\|_B\Big| \leq \epsilon, \ \sum_{k\geq p} \|u_k^{(n)}\|_V \leq \epsilon. \tag{1.73}$$

Then in fact we can choose successive $p_j > p_{j-1}$ (remember that little n is fixed here) so that

$$\Big|\|\sum_{k\leq p_j} u_k^{(n)}\|_V - \|b_n\|_B\Big| \leq 2^{-j}\epsilon, \ \sum_{k\geq p_j} \|u_k^{(n)}\|_V \leq 2^{-j}\epsilon \ \forall\ j. \tag{1.74}$$

Now, 'resum the series' defining instead $v_1^{(n)} = \sum_{k=1}^{p_1} u_k^{(n)}$, $v_j^{(n)} = \sum_{k=p_{j-1}+1}^{p_j} u_k^{(n)}$ and do this setting $\epsilon = 2^{-n}$ for the nth series. Check that now

$$\sum_n \sum_k \|v_k^{(n)}\|_V < \infty. \tag{1.75}$$

Of course, you should also check that $b_n = \{v_k^{(n)}\} + S$ so that these new summable series work just as well as the old ones.

After this fiddling you can now try to find a limit for the sequence as

$$b = \{w_k\} + S,\ w_k = \sum_{l+p=k+1} v_l^{(p)} \in V. \tag{1.76}$$

So, you need to check that this $\{w_k\}$ is absolutely summable in V and that $b_n \to b$ as $n \to \infty$.

Finally then there is the question of showing that $I(V)$ is dense in B. You can do this using the same idea as above – in fact it might be better to do it first. Given an element $b \in B$ we need to find elements in V, v_k such that $\|I(v_k) - b\|_B \to 0$ as $k \to \infty$. Take an absolutely summable series u_k representing b and take $v_j = \sum_{k=1}^{N_j} u_k$ where the p_j's are constructed as above and check that $I(v_j) \to b$ by computing

$$\|I(v_j) - b\|_B = \lim_{\to\infty} \|\sum_{k>p_j} u_k\|_V \leq \sum_{k>p_j} \|u_k\|_V. \tag{1.77}$$

- c_0 the space of convergent sequences in C with supremum norm, a Banach space.
- l^p one space for each real number $1 \leq p < \infty$; the space of p-summable series with corresponding norm; all Banach spaces. The most important of these for us is the case $p = 2$, which is (a) Hilbert space.
- l^∞ the space of bounded sequences with supremum norm, a Banach space with $c_0 \subset l^\infty$ as a closed subspace with the same norm.
- C([a,b]) or more generally C(M) for any compact metric space M – the Banach space of continuous functions with supremum norm.
- C_∞(R), or more generally $C_\infty(M)$ for any metric space M – the Banach space of bounded continuous functions with supremum norm.

- C_0(R), or more generally $C_0(M)$ for any metric space M – the Banach space of continuous functions which 'vanish at infinity' (see Problem 1) with supremum norm. A closed subspace, with the same norm, in C_∞^0 (M).
- C^k([a,b]) the space of k times continuously differentiable (so $k \in$ N) functions on [a,b] with norm the sum of the supremum norms on the function and its derivatives. Each is a Banach space – see Problem 1.
- The space C([0,1]) with norm

(1.78)
$$\|u\|_{L^1} = \int_0^1 |u| dx$$

given by the Riemann integral of the absolute value. A normed space, but not a Banach space. We will construct the concrete completion, L^1([0,1]) of Lebesgue integrable 'functions'.
- The space R([a,b]) of Riemann integrable functions on [a,b] with kuk defined by (1.78). This is only a seminorm, since there are Riemann integrable functions (note that u Riemann integrable does imply that $|u|$ is Riemann integrable) with $|u|$ having vanishing Riemann integral but which are not identically zero. This cannot happen for continuous functions. So the quotient is a normed space, but it is not complete.
- The same spaces – either of continuous or of Riemann integrable functions but with the (semi- in the second case) norm

(1.79)
$$\|u\|_{L^p} = \left(\int_a^b |u|^p\right)^{\frac{1}{p}}.$$

Not complete in either case even after passing to the quotient to get a norm for Riemann integrable functions. We can, and indeed will, define $L^p(a,b)$ as the completion of C([a,b]) with respect to the L^p norm. However we will get a concrete realization of it soon.
- Suppose $0 < \alpha < 1$ and consider the subspace of C([a,b]) consisting of the 'H¨older continuous functions' with exponent α, that is those $u : [a,b] \longrightarrow$ C which satisfy

(1.80) $$|u(x) - u(y)| \leq C|x-y|^{\alpha} \text{ for some } C \geq 0.$$

Note that this already implies the continuity of u. As norm one can take the sum of the supremum norm and the 'best constant' which is the same as

(1.81) $$\|u\|_{C^{\alpha}} = \sup_{x\in[a,b]|} |u(x)| + \sup_{x\neq y\in[a,b]} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}};$$

it is a Banach space usually denoted $C^{\alpha}([a,b])$.

- Note the previous example works for $\alpha = 1$ as well, then it is not denoted $C^1([a,b])$, since that is the space of once continuously differentiable functions; this is the space of Lipschitz functions – again it is a Banach space.
- We will also talk about Sobolev spaces later. These are functions with 'Lebesgue integrable derivatives'. It is perhaps not easy to see how to define these, but if one takes the norm on $C^1([a,b])$

(1.82) $$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\frac{du}{dx}\|_{L^2}^2\right)^{\frac{1}{2}}$$

and completes it, one gets the Sobolev space $H^1([a,b])$ – it is a Banach space (and a Hilbert space). In fact it is a subspace of $C([a,b]) = C^0([a,b])$.

Here is an example to see that the space of continuous functions on [0,1] with norm (1.78) is not complete; things are even worse than this example indicates! It is a bit harder to show that the quotient of the Riemann integrable functions is not complete, feel free to give it a try.

Take a simple non-negative continuous function on $\mathbb{R}$ for instance

(1.83) $$f(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then $\int_{-1}^{1} f(x) = 1$. Now scale it up and in by setting

(1.84) $$f_N(x) = Nf(N^3x) = 0 \text{ if } |x| > N^{-3}.$$

So it vanishes outside $[-N^{-3},N^{-3}]$ and has $\int_{-1}^{1} f_N(x)dx = N^{-2}$. It follows that the sequence $\{f_N\}$ is absolutely summable with respect to the integral norm in (1.78) on $[-1,1]$. The pointwise series $\sum_N f_N(x)$ converges everywhere except at $x = 0$ – since at each point $x \neq 0$, $f_N(x) = 0$ if $N^3|x| > 1$. The resulting function, even if we ignore the problem at $x = 0$, is not Riemann integrable because it is not bounded.

Linear spaces and the Hahn Banach Theorem

Many objects in mathematics — particularly in analysis — are, or may be described in terms of, linear spaces (also called vector spaces). For example:

(1) $C(M)$ = space of continuous functions (R or C valued) on a manifold M.

(2) $A(U)$ = space of analytic functions in a domain $U \subset$ C.

(3) $L^p(\mu)$ = {p integrable functions on a measure space M,μ}.

The key features here are the axioms of linear algebra,

Definition 1.1. A linear space X over a field F (in this course F = R or C) is a set on which we have defined

(1) addition: $x,y \in X \mapsto x + y \in X$

and

(2) scalar multiplication: $k \in F$, $x \in X \mapsto kx \in X$

with the following properties

(1) $(X,+)$ is an abelian group (+ is commutative and associative and ∃ identity and inverses.)
- identity is called 0 (“zero”)
- inverse of x is denoted $-x$

(2) scalar multiplication is
- associative: $a(bx) = (ab)x$,
- distributive: $a(x + y) = ax + by$ and $(a + b)x = ax + bx$, and satisfies $1x = x$.

Remark. It follows from the axioms that $0x = 0$ and $-x = (-1)x$.

Recall from linear algebra that a set of vectors $S \subset X$ is linearly independent if

$$\sum_{j=1}^{n} a_j x_j = 0 \text{ with } x_1,\ldots,x_n \in S \implies a_1 = \cdots = a_n = 0$$

and that the dimension of X is the cardinality of a maximal linearly independent set in X. The dimension is also the cardinality of a minimal spanning set, where the span of a set S is the set

$$S = \left\{ \sum_{j=1}^{n} a_j x_j \ : \ a_1, \ldots, a_n \in \mathbb{R} \right)$$

spanand $x_1,\ldots,x_n \in S$

,

and S is spanning, or spans X, if span$S = X$.

More or less, functional analysis is linear algebra done on spaces with infinite dimension. Stated this way it may seem odd that functional analysis is part of analysis. For finite

dimensional spaces the axioms of linear algebra are very rigid: there is essentially only

one interesting topology on a finite dimensional space and up to isomorphism there is only one linear space of each finite dimension. In infinite dimensions we shall see that topology matters a great deal, and the topologies of interest are related to the sort of analysis that one is trying to do.

That explains the second word in the name ”functional analysis.” Regarding “functional,” this is an archaic term for a function defined on a domain of functions. Since most of the spaces we study are function spaces, like $C(M)$, the functions defined on them are “functionals.” Thus “functional analysis.” In particular, we define a linear functional to be a linear map $` : X \to F$, which means

$$`(x + y) = `(x) + `(y) \text{ and } `(ax) = a`(x) \text{ for all } x,y \in X \text{ and } a \in F.$$

Often, one is able to define a linear functional at first only for a limited set of vectors $Y \subset X$. For example, one may define the Riemann integral on $Y = C[0,1]$, say, which is a subset of the space $B[0,1]$ of all bounded functions on $[0,1]$. In most cases, as in the example, the set Y is a subspace:

Definition 1.2. A subset $Y \subset X$ of a linear space is a linear subspace if it is closed

under addition and scalar multiplication: $y_1,y_2 \in Y$ and $a \in F \Longrightarrow y_1 + ay_2 \in Y$.

For functionals defined, at first, on a subspace of a linear space of R we have

Theorem 1.1 (Hahn (1927), Banach (1929)). *Let X be a linear space over* R *and p a real valued function on X with the properties*

(1) $p(ax) = ap(x)$ for all $x \in X$ and $a > 0$ (Positive homogeneity)
(2) $p(x + y) \le p(x) + p(y)$ for all $x,y \in X$ (subadditivity).

If ` is a linear functional defined on a linear subspace of $Y$ and dominated by $p$, that is $`(y) \le p(y)$ for all $y \in Y$, then ` can be extended to all of $X$ as a linear functional dominated by $p$, so $`(x) \le p(x)$ for all $x \in X$.

Example. Let $X = B[0,1]$ and $Y = C[0,1]$. On Y, let $\ell(f) = \int_0^1 f(t)\mathrm{d}t$ (Riemann integral). Let $p : B \to$ R be $p(f) = \sup\{|f(x)| : x \in [0,1]\}$. Then p satisfies (1) and (2) and $`(f) \le p(f)$. Thus we can extend ` to all of $B[0,1]$. We will return to this example and see that we can extend ` so that $`(f) \ge 0$ whenever $f \ge 0$. This defines a finitely additive set function on <u>all</u>(!) subset of $[0,1]$ via $\mu(S) = \ell(\chi_S)$. For Borel measurable sets it turns out the result is Lebesgue measure. That does not follow from Hahn-Banach however.

The proof of Hahn-Banach is <u>not</u> constructive, but relies on the following result equivalent to the axiom of choice:

Theorem 1.2 (Zorn's Lemma). *Let S be a partially ordered set such that every totally ordered subset has an upper bound. Then S has a maximal element.*

To understand the statement, we need

Definition 1.3. A <u>partially ordered set</u> S is a set on which an order relation $a \leq b$ is defined for some (but not necessarily all) pairs $a,b \in S$ with the following properties

(1) transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$
(2) reflexivity: if $a \leq a$ for all $a \in S$.

(Note that (1) asserts two things: that a and c are comparable and that $a \leq c$.) A subset T of S is <u>totally ordered</u> if $x,y \in T \implies x \leq y$ or $y \leq x$. An element $u \in S$ is an <u>upper bound</u> for $T \subset S$ if $x \in T \implies x \leq u$. A <u>maximal element</u> $m \in S$ satisfies $m \leq b \implies m = b$.

Proof of Hahn-Banach. To apply Zorn's Lemma, we need a poset, $S =$

$$\{\text{extensions of } \ell \text{ dominated by } p\}.$$

That is S consists of pairs (ℓ', Y') with ℓ' a linear functional defined on a subspace $Y' \supset Y$ so that

$$\ell'(y) = \ell(y),\ y \in Y \text{ and } \ell'(y) \leq p(y),\ y \in Y'.$$ Order S as follows

$$(\ell_1, Y_1) \leq (\ell_2, Y_2) \iff Y_1 \subset Y_2 \text{ and } \ell_2|_{Y_1} = \ell_1.$$

If T is a totally ordered subset of S, let $(\bar{\ell}, \bar{Y})$ be

$$\bar{Y} = \bigcup \{Y' : (\ell', Y') \in T\}$$

and

$$\bar{\ell}(y) = \ell'(y) \text{ for } y \in Y'.$$

Since T is totally ordered the definition of $\bar{\ell}$ is unambiguous. Clearly $(\bar{\ell}, \bar{Y})$ is an upper bound for T. Thus by Zorn's Lemma there exists a maximal element (ℓ^+, Y^+).

To finish, we need to see that $Y^+ = X$. It suffices to show that (ℓ', Y') has an extension whenever $Y' \neq X$. Indeed, let $x_0 \in X$. We want ℓ'' on $Y'' = \{ax_0 + y : y \in Y, a \in \mathbb{R}\}$. By linearity we need only define $\ell''(x_0)$. The constraint is that we need

$$a\ell''(x_0) + \ell'(y) \leq p(ax_0 + y)$$

for all a,y. Dividing through by $|a|$, since Y^0 is a subspace, we need only show

$$\pm\ `^{00}(x_0) \leq p(y \pm x_0) - `^0(y) \tag{1.1}$$

for all $y \in Y^0$. We can find $`^{00}(x_0)$ as long as

$$`^0(y^0) - p(y^0 - x_0) \leq p(x_0 + y) - `^0(y) \text{ for all } y,y^0 \in Y^0, \tag{1.2}$$

or equivalently

$$`^0(y^0 + y) \leq p(x_0 + y) + p(y^0 - x_0) \text{ for all } y,y^0 \in Y^0. \tag{1.3}$$

Since

$$`^0(y^0 + y) \leq p(y^0 + y) = p(y^0 - x_0 + y + x_0) \leq p(x_0 + y) + p(y^0 - x_0),$$

(1.3), and thus (1.2), holds. So we can satisfy (1.1).

In finite dimensions, one can give a constructive proof involving only finitely many choices. In infinite dimensions the situation is a quite a bit different, and the approach via Zorn's lemma typically involves uncountably many "choices."

Geometric Hahn-Banach Theorems

We may use Hahn-Banach to understand something of the geometry of linear spaces. We want to understand if the following picture holds in infinite dimension:

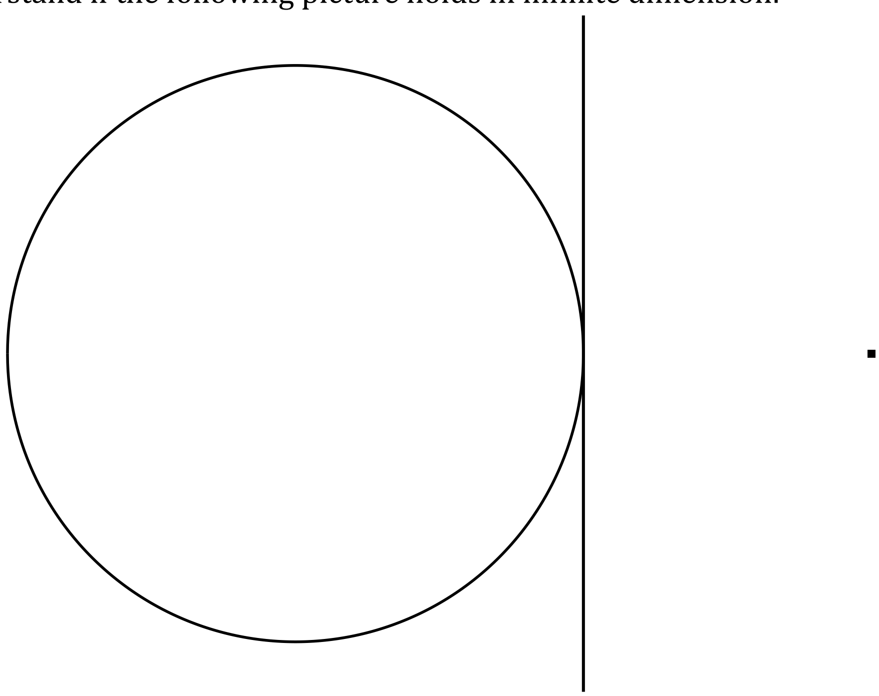

Figure 2.1. Separating a point from a convex set by a ~~line~~ hyperplane

Definition 2.1. A set $S \subset X$ is <u>convex</u> if for all $x,y \in S$ and $t \in [0,1]$ we have $tx + (1 - t)y \in S$.

Definition 2.2. A point $x \in S \subset X$ is an interior point of S if for all $y \in X$ $\exists \varepsilon > 0$ s.t.

$|t| < \varepsilon \Longrightarrow x + ty \in S$.

Remark. We can a define a topology using this notion, letting $U \subset X$ be open $\Longleftrightarrow$ all $x \in U$ are interior. From the standpoint of abstract linear algebra this seems to be a "natural" topology on X. In practice, however, it has way too many open sets and we work with weaker topologies that are relevant to the analysis under considerations. Much of functional analysis centers around the interplay of different topologies.

We are aiming at the following

Theorem 2.1. *Let K be a non-empty convex subset of X, a linear space over $\mathbb{R}$, and suppose K has at least one interior point. If $y \notin K$ then $\exists$ a linear functional $\ell : X \to \mathbb{R}$ s.t.*

$$\ell(x) \leq \ell(y) \text{ for all } x \in K, \tag{2.1}$$

with strict inequality for all interior points x of K.

This is the "hyperplane separation theorem," essentially validates the picture drawn above. A set of the form $\{\ell(x) = c\}$ with ℓ a linear functional is a "hyperplane" and the sets $\{\ell(x) < c\}$ are "half spaces."

To accomplish the proof we will use Hahn-Banach. We need a dominating function p.

Definition 2.3. Let $K \subset X$ be convex and suppose 0 is an interior point. The gauge of K (with respect to the origin) is the function $p_K : X \to \mathbb{R}$ defined as

$$p_K(x) = \inf\{a : a > 0 \text{ and } \frac{x}{a} \in K\}.$$

(Note that $p_K(x) < \infty$ for all x since 0 is interior.)

Lemma 2.2. *p_K is positive homogeneous and sub-additive.*

Proof. Positive homogeneity is clear (even if K is not convex). To prove sub-additivity we use convexity. Consider $p_K(x + y)$. Let a,b be such that $x/a, y/b \in K$. Then

$$t\frac{x}{a} + (1-t)\frac{y}{b} \in K \quad \forall t \in [0,1],$$

so

$$\frac{x+y}{a+b} = \frac{a}{a+b}\frac{x}{a} + \frac{b}{a+b}\frac{y}{b} \in K.$$

Thus $p_K(x + y) \leq a + b$, and optimizing over a,b we obtain the result.

Proof of hyperplane separation thm. Suffices to assume $0 \in K$ is interior and $c = 1$. Let p_K be the gauge of K. Note that $p_K(x) \leq 1$ for $x \in K$ and that $p_K(x) < 1$ if x is interior, as then $(1 + t)x \in K$ for small $t > 0$. Conversely if $p_K(x) < 1$ then x is an interior point (why?), so

$$p_K(x) < 1 \iff x \in K.$$

Now define $\ell(y) = 1$, so $\ell(ay) = a$ for $a \in \mathbb{R}$. Since $y \notin K$ it is not an interior point and so $p_K(y) \geq 1$. Thus $p_K(ay) \geq a$ for $a \geq 0$ and also, trivially, for $a < 0$ (since $p_K \geq 0$). Thus

$$p_K(ay) \geq \ell(ay)$$

for all $a \in \mathbb{R}$. By Hahn-Banach, with Y the one dimensional space $\{ay\}$, ℓ may be extended to all of x so that $p_K(x) \geq \ell(x)$ which implies (2.1).

An extension of this is the following

Theorem 2.3. *Let H,M be disjoint convex subsets of X, at least one of which has an interior point. Then H and M can be separate by a hyperplane $\ell(x) = c$: there is a linear functional ℓ and $c \in \mathbb{R}$ such that*

$$\ell(u) \leq c \leq \ell(v) \forall u \in H, v \in M.$$

Proof. The proof rests on a trick of applying the hyperplane separation theorem with the set

$$K = H - M = \{u - v : u \in H \text{ and } v \in M\}$$

and the point $y = 0$. Note that $0 \notin K$ since $H \cup M = \emptyset$. Since K has an interior point (why?), we see that there is a linear functional such that $\ell(x) \leq 0$ for all $x \in K$. But then
$\ell(u) \leq \ell(v)$ for all $u \in H$, $v \in M$.

In many applications, one wants to consider a vector space X over $\mathbb{C}$. Of course, then X is also a vector space over $\mathbb{R}$ so the real Hahn-Banach theorem applies. Using this one can show the following

Theorem 2.4 (Complex Hahn-Banach, Bohenblust and Sobczyk (1938) and Soukhomlinoff (1938)). *Let X be a linear space over $\mathbb{C}$ and $p : X \to [0,\infty)$ such that*

(1) $p(ax) = |a|p(x) \forall a \in \mathbb{C}, x \in X$.
(2) $p(x + y) \leq p(x) + p(y)$ (sub-additivity).

Let Y be a $\mathbb{C}$ linear subspace of X and $\ell : Y \to \mathbb{C}$ a linear functional such that

$$|\ell(y)| \leq p(y) \tag{2.2}$$

for all $y \in Y$. Then ℓ can be extended to all of X so that (2.2) *holds for all $y \in X$.*

Remark. A function p that satisfies (1) and (2) is called a <u>semi-norm</u>. It is a <u>norm</u> if $p(x) = 0 \implies x = 0$.

Proof. Let $\ell_1(y) = \mathrm{Re}\,\ell(y)$, the real part of ℓ. Then ℓ_1 is a real linear functional and $-\ell_1(iy) = -\mathrm{Re}\,i\ell(y) = \mathrm{Im}\,\ell(y)$, the imaginary part of ℓ. Thus

$$\ell(y) = \ell_1(y) - i\ell_1(iy). \tag{2.3}$$

Clearly $|\ell_1(y)| \le p(y)$ so by the real Hahn-Banach theorem we can extend ℓ_1 to all of X so that $\ell_1(y) \le p(y)$ for all $y \in X$. Since $-\ell_1(y) = \ell_1(-y) \le p(-y) = p(y)$, we have $|\ell_1(y)| \le p(y)$ for all $y \in X$. Now define the extension of ℓ via (2.3). Given $y \in X$ let $\theta = \operatorname{argln}\ell(y)$. Thus $\ell(y) = e^{i\theta}\ell_1(e^{-i\theta}y)$ (why?). So,

$$|\ell(y)| = |\ell_1(e^{-i\theta}y)| \le p(y).$$

Lax gives another beautiful extension of Hahn-Banach, due to Agnew and Morse, which involves a family of commuting linear maps. We will cover a simplified version of this next time.

Applications of Hahn-Banach

To get an idea what one can do with the Hahn-Banach theorem let's consider a concrete application on the linear space $X = B(S)$ of all real valued bounded functions on a set S. $B(S)$ has a natural partial order, namely $x \le y$ if $x(s) \le y(s)$ for all $s \in S$. If $0 \le x$ then x is nonnegative. On $B(S)$ a positive linear functional ℓ satisfies $\ell(y) \ge 0$ for all $y \ge 0$.

Theorem 3.1. *Let Y be a linear subspace of $B(S)$ that contains $y_0 \ge 1$, so $y_0(s) \ge 1$ for all $s \in S$. If ℓ is a positive linear functional on Y then ℓ can be extended to all of B as a positive linear functional.*

This theorem can be formulated in an abstract context as follows. The nonnegative functions form a <u>cone</u>, where

Definition 3.1. A subset $P \subset X$ of a linear space over $\mathbb{R}$ is a <u>cone</u> if $tx + sy \in P$ whenever $x,y \in P$ and $t,s \ge 0$. A linear functional on X is P-nonnegative if $\ell(x) \ge 0$ for all $x \in P$.

Theorem 3.2. *Let $P \subset X$ be a cone with an interior point x_0. If Y is a subspace containing x_0 on which is defined a $P \cap Y$-positive linear functional ℓ, then ℓ has an extension to X which is P-positive.*

Proof. Define a dominating function p as follows

$$p(x) = \inf\{\ell(y) : y - x \in P, y \in Y\}.$$

Note that $y_0 - tx \in P$ for some $t > 0$ (since y_0 is interior to P), so $\frac{1}{t}y_0 - x \in P$. This shows that $p(x)$ is well defined. It is clear that p is positive homegeneous. To see that it is sub-additive, let $x_1,x_2 \in X$ and let y_1,y_2 be so that $y_j - x_j \in P$. Then $y_1+y_2-(x_1+x_2) \in P$, so

$$p(x_1 + x_2) \le \ell(y_1) + \ell(y_2).$$

Minimizing over $y_{1,2}$ gives sub-additivity.

Since $\ell(x) = \ell(x-y)+\ell(y) \le \ell(y)$ if $x \in Y$ and $y - x \in P$ we conclude that $\ell(x) \le p(x)$ for all $x \in Y$. By Hahn-Banach we may extend ℓ to all of X so that $\ell(x) \le p(x)$ for all x.

Now let $x \in P$. Then $p(-x) \le 0$ (why?), so $-\ell(x) = \ell(-x) \le 0$ which shows that ℓ is P-positive.

The theorem on $B(S)$ follows from the Theorem 3.2 once we observe that the condition $y_0 \geq 1$ implies that y_0 is an interior point of the cone of positive functions. The linear functional that one constructs in this way is <u>monotone</u>:

$$x \leq y \quad \Longrightarrow \quad \ell(x) \leq \ell(y),$$

which is in fact equivalent to positivity. This can allow us to do a little bit of analysis, even though we haven't introduced notions of topology or convergence.

To see how this could work, let's apply the result to the Riemann integral on $C[0,1]$. We conclude the existence of a positive linear functional $\ell : B[0,1] \to \mathbb{R}$ that gives $\ell(f) = \int_0^1 f(x)\mathrm{d}x$. This gives an "integral" of arbitrary bounded functions, <u>without a measurability condition</u>. Since the integral is a linear functional, it is finitely additive. Of course it is not countably additive since we know from real analysis that such a thing doesn't exist. Furthermore, the integral is not uniquely defined.

Nonetheless, the extension is also not arbitrary, and the constraint of positivity actually pins down $\ell(f)$ for many functions. For example, consider $\chi_{(a,b)}$ (the characteristic function of an open interval). By positivity we know that

$$f \leq \chi_{(a,b)} \leq g,\ f,g \in C[0,1] \Longrightarrow \int_0^1 f(x)\mathrm{d}x \leq \ell(\chi_{(a,b)}) \leq \int_0^1 g(x)\mathrm{d}x.$$

Taking the sup over f and inf over g, using properties only of the Riemann integral, we see that $\ell(\chi_{(a,b)}) = b - a$ (hardly a surprising result). Likewise, we can see that $\ell(\chi_U) = |U|$ for any open set, and by finite additivity $\ell(\chi_F) = 1 - \ell(F^c) = 1 - |F^c| = |F|$ for any closed set ($|\cdot|$ is Lebesgue measure). Finally if $S \subset [0,1]$ and

$$\sup\{\ell(F) : F \text{ closed and } F \subset S\} = \inf\{\ell(U) : U \text{ open and } U \supset S\}$$

then $\ell(\chi_S)$ must be equal to these two numbers. We see that $\ell(\chi_S) = |S|$ for any Lebesgue measurable set. We have just painlessly constructed Lebesgue measure from the Riemann integral without using any measure theory!

The rigidity of the extension is a bit surprising if we compare with what happens in finite dimensions. For instance, consider the linear functional $\ell(x,0) = x$ defined on $Y = \{(x,0)\} \subset \mathbb{R}^2$. Let P be the cone $\{(x,y) : |y| \leq \alpha x\}$. So ℓ is $P \cap Y$-positive. To extend ℓ to all of $\mathbb{R}^2$ we need to define $\ell(0,1)$. To keep the extension positive we need only require

$$y\ell(0,1) + 1 \geq 0$$

if $|y| \leq \alpha$. Thus we must have $|\ell(0,1)| \leq 1/\alpha$, and any choice in this interval will work. Only when $\alpha = \infty$ and the cone degenerates to a half space does the condition pin $\ell(0,1)$ down. So some interesting things happen in ∞ dimensions.

A second example application assigning a limiting value to sequences

$$\mathbf{a} = (a_1, a_2, \ldots).$$

Let B denote the space of all bounded R-valued sequences and let L denote the subspace of sequences with a limit. We quickly conclude that there is an positive linear extension of the positive linear functional lim to all of B. We would like to conclude a little more, however. After all, for convergent sequences,

$$\lim a_{n+k} = \lim a_n$$

for any k.

To formalize this property we define a linear map on B by

$$T(a_1,a_2,...) = (a_2,a_3,...).$$

(T is the "backwards shift" operator or "left translation.") Here,

Definition 3.2. Let X_1,X_2 be linear spaces over a field F. A linear map $T : X_1 \to X_2$

is a function such that

$$T(x + ay) = T(x) + aT(y) \qquad \forall x,y \in X_1 \text{ and } a \in F.$$

Normed and Banach Spaces

The Hahn-Banach theorem made use of a dominating function $p(x)$. When this function is non-negative, it can be understood roughly as a kind of "distance" from a point x to the origin. For that to work, we should have $p(x) > 0$ whenever x 6= 0. Such a function is called a norm:

Definition 4.1. Let X be a linear space over F = R or C. A norm on X is a function k·k : $X \to [0,\infty)$ such that

(1) kxk = 0 $\iff x = 0$.
(2) k$x + y$k ≤ kxk + kyk (subadditivity)
(3) kaxk = $|a|$kxk for all $a \in F$ and $x \in X$ (homogeneity).

A normed space is a linear space X with a norm k·k.

The norm on a normed space gives a metric topology if we define the distance between two points via

$$d(x,y) = \text{k}x - y\text{k}.$$

Condition 1 guarantees that two distinct points are a finite distance apart. Sub-additivity gives the triangle inequality. The metric is

(1) translation invariant: $d(x + z,y + z) = d(x,y)$
and
(2) homogeneous $d(ax,ay) = |a|d(x,y)$.

Thus any normed space is a metric space and we have the following notions:

(1) a sequence x_n converges to x, $x_n \to x$, if $d(x_n,x)$ = k$x_n - x$k $\to 0$.
(2) a set $U \subset X$ is open if for every $x \in U$ there is a ball $\{y \ : \ \|y - x\| < \epsilon\} \subset U$.

(3) a set $K \subset X$ is closed if $X \setminus K$ is open.
(4) a set $K \subset X$ is ~~compact if~~ every open cover of K has a finite sub-cover.

The norm defines the topology but not the other way around. Indeed two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if there is $c > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq c^{-1}\|x\|_2 \qquad \forall x \in X.$$

Equivalent norms define the same topology. (Why?)

Recall from real analysis that a metric space X is complete if every Cauchy sequence x_n converges in X. In a normed space, a Cauchy sequence x_n is one such that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } n, m > N \implies \|x_n - x_m\| < \epsilon.$$

A complete normed space is called a Banach space.

Not every normed space is complete. For example $C[0,1]$ with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\,dx$$

fails to be complete. (It is, however, a complete space in the uniform norm, $\|f\|_u = \sup_{x\in[0,1]} |f(x)|$.) However, every normed space X has a completion, defined abstractly as a set of equivalence classes of Cauchy sequences in X. This space, denoted $\overline{X}$ is a Banach space.

Examples of Normed and Banach spaces

(1) For each $p \in [1,\infty)$ let

$$\ell_p = \{p \text{ summable sequences}\} = \{(a_1,a_2,\ldots) \mid \sum_{j=1}^{\infty} |a_j|^p < \infty\}.$$

Define a norm on ℓ_p via

$$\|\mathbf{a}\|_p = \left[\sum_{j=1}^{\infty} |a_j|^p\right]^{\frac{1}{p}}.$$

Then ℓ_p is a Banach space.

(2) Let

$$\ell_\infty = \{\text{bounded sequences}\} = B(\mathbb{N}),$$

with norm

$$\|\mathbf{a}\|_\infty = \sup_j |a_j|. \qquad (?)$$

Then ℓ_∞ is a Banach space.

(3) Let

$$c_0 = \{\text{sequences converging to } 0\} = \{(a_1,a_2,\ldots) \mid \lim_{j\to\infty} a_j = 0\},$$

with norm (?). Then c_0 is a Banach space.

(4) Let

$$\begin{aligned} \mathrm{F} &= \{\text{sequences with finitely many non-zero terms}\} \\ &= \{(a_1,a_2,\ldots) \mid \exists N \in \mathbb{N} \text{ such that } n \geq N \implies a_n = 0\}. \end{aligned}$$

Then for any $p \geq 1$, $\mathrm{F}_p = (\mathrm{F}, \|\cdot\|_p)$ is a normed space which is not complete. The completion of F_p is isomorphic to ℓ_p.

(5) Let $D \subset \mathbb{R}^d$ be a domain and let $p \in [1,\infty)$.

(a) Let $X = C_c(D)$ be the space of continuous functions with compact support in D, with the norm

$$\|f\|_p = \left[\int_D |f(x)|^p \mathrm{d}x\right]^{\frac{1}{p}}.$$

Then X is a normed space, which is not complete. Its completion is denoted $L^p(D)$ and may be identified with the set of equivalence classes of measurable functions $f : D \to \mathbb{C}$ such that

$$\int_D |f(x)|^p \mathrm{d}x < \infty \qquad \text{(Lebesgue measure)},$$

with two functions f,g called equivalent if $f(x) = g(x)$ for almost every x.

(b) Let X denote the set of C^1 functions on D such that

$$\int_D |f(x)|^p \mathrm{d}x < \infty \quad \text{and} \quad \int_D |\partial_j f(x)|^p \mathrm{d}x < \infty,\ j = 1,\ldots,n.$$

Put the following norm on X,

$$\|f\|_{1,p} = \left[\int_D |f(x)|^p \mathrm{d}x + \sum_{j=1}^{n} \int_D |\partial_j f(x)|^p \mathrm{d}x.\right]^{\frac{1}{p}}.$$

Then X is a normed space which is not complete. Its completion is denoted $W^{1,p}(D)$ and is called a Sobolev space and may be identified with the subspace of $L^p(D)$ consisting of (equivalence classes) of functions all of whose first derivatives are in $L^p(D)$ in the sense of distributions. (We'll come back to this.) Note that $\mathbf{a} \in \ell_1$ is summable,

$$\sum_j a_j \le \|\mathbf{a}\|_1.$$

Theorem 4.1 (Hölder's Inqequality). *Let* $1 < p < \infty$ *and let* q *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $\mathbf{a} \in \ell_p$ *and* $\mathbf{b} \in \ell_q$ *then* $\mathbf{ab} = (a_1b_1, a_2b_2, \ldots) \in \ell_1$ *and*

$$\left|\sum_{j=1}^{\infty} a_j b_j\right| \le \|\mathbf{a}\|_p \|\mathbf{b}\|_q.$$

Proof. First note that for two non-negative numbers a,b it holds that

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

For $p = q = 2$ this is the familiar "arithmetic-geometric mean" inequality which follows since $(a-b)^2 \ge 0$. For general a,b this may be seen as follows. The function $x \mapsto \exp(x)$ is convex: $\exp(tx + (1-t)y) \le t\exp(x) + (1-t)\exp(y)$. (Recall from calculus that a C^2 function f is convex if $f'' \ge 0$.) Thus,

$$ab = \exp(\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q) \le \frac{1}{p}\exp(\ln a^p) + \frac{1}{q}\exp(\ln b^q) = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Applying this bound co-ordinate wise and summing up we find that

$$\|\mathbf{ab}\|_1 \le \frac{1}{p}\|\mathbf{a}\|_p^p + \frac{1}{q}\|\mathbf{b}\|_q^q.$$

The result follows from this bound by "homogenization:" we have

$$\|\mathbf{a}\|_p, \|\mathbf{b}\|_q \implies \|\mathbf{ab}\| \le 1,$$

so $\left\| \frac{\mathbf{a}}{\|\mathbf{a}\|_p} \frac{\mathbf{b}}{\|\mathbf{b}\|_q} \right\|_1 \le 1$ from which the desired estimate follows by homogeneity of the norm.

Similarly, we have

Theorem 4.2 (Hölder's Inequality). *Let* $1 < p < \infty$ *and let* q *be the conjugate exponent. If* $f \in L^p(D)$ *and* $g \in L^q(D)$ *then* $fg \in L^1(D)$ *and*

$$\int_D |f(x)g(x)|\mathrm{d}x \le \|f\|_p \|g\|_q.$$

Noncompactness of the Ball and Uniform Convexity

First a few more definitions:

Definition 5.1. A normed space X over $F = \mathbb{R}$ or $\mathbb{C}$ is called separable if it has a countable, dense subset.

Most spaces we consider are separable, with a few notable exceptions.

(1) The space M of all signed (or complex) measures μ on $[0,1]$, say, with norm

$$\|\mu\| = \int_0^1 |\mu|(\mathrm{d}x).$$

Here $|\mu|$ denotes the total variation of μ,

$$|\mu|(A) = \sup_{\substack{\text{Partitions} \\ A_1,\ldots,A_n \text{ of } A}} \sum_{j=1}^{n} |\mu(A_j)| .$$

This space is a Banach space. Since the point mass

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is an element of A and $\|\delta_x - \delta_y\| = 2$ if $x \neq y$, we have an uncountable family of elements of M all at a fixed distance of one another. Thus there can be no countable dense subset. (Why?)

(2) ℓ_∞ is also not separable. To see this, note that to each subset of $A \subset \mathbb{N}$ we may associate the sequence χ_A, and

$$\|\chi_A - \chi_B\|_\infty = 1$$

if $A \neq B$.

(3) ℓ_p is separable for $1 \le p < \infty$.

(4) $L^p(D)$ is separable for $1 \le p < \infty$.

(5) $L^\infty(D)$ is not separable.

Noncompactness of the Unit Ball

Theorem 5.1 (F. Riesz). *Let X be a normed linear space. Then the closed unit ball $B_1(0) = \{x : \|X\| \le 1\}$ is compact if and only if X is finite dimensional.*

Proof. The fact that the unit ball is compact if X is finite dimensional is the HeineBorel Theorem from Real Analysis. To see the converse, we use the following

Lemma 5.2. *Let Y be a closed proper subspace of a normed space X. Then there is a unit vector $z \in X$,* kzk = 1*, such that*

$$\|z - y\| > \frac{1}{2} \quad \forall y \in Y.$$

Proof of Lemma. Since Y is proper, $\exists x \in X \setminus Y$. Then

$$\inf_{y \in Y} \|x - y\| = d > 0.$$

(This is a property of closed sets in a metric space.) We do not know the existence of a minimizing y, but we can certainly find y_0 such that

$$0 < kx - y_0k < 2d.$$

Let $z = \frac{x - y_0}{\|x - y_0\|}$. Then

$$\|z - y\| \;=\; \frac{\|x - y_0 - \|x - y_0\|\, y\|}{\|x - y_0\|} \geq \frac{d}{2d} = \frac{1}{2}.$$

Returning to the proof of the Theorem, we will use the fact that every sequence in a compact metric space has a convergent subsequence. Thus it suffices to show that if X is infinite dimensional then there is a sequence in $B_1(0)$ with no convergent subsequence.

Let y_1 be any unit vector and construct a sequence of unit vectors, by induction, so that

$$\|y_k - y\| > \frac{1}{2} \quad \forall y \in \operatorname{span}\{y_1, \dots, y_{k-1}\}.$$

(Note that $\operatorname{span}\{y_1,\dots,y_{k-1}\}$ is finite dimensional, hence complete, and thus a closed subspace of X.) Since X is finite dimensional the process never stops. No subsequence of y_j can be Cauchy, much less convergent.

Uniform convexity

The following theorem may be easily shown using compactness:

Theorem 5.3. *Let X be a finite dimensional linear normed space. Let K be a closed convex subset of X and z any point of X. Then there is a unique point of K closer to z than any other point of K. That is there is a unique solution $y_0 \in K$ to the minimization problem*

$$\|y_0 - z\| = \inf_{y \in K} \|y - z\|. \tag{?}$$

Try to prove this theorem. (The existence of a minimizer follows from compactness; the uniqueness follows from convexity.)

The conclusion of theorem does not hold in a general infinite dimensional space. Nonetheless there is a property which allows for the conclusion, even though compactness fails!

Definition 5.2. A normed linear space X is uniformly convex if there is a function

$\epsilon : (0,\infty) \to (0,\infty)$, such that

(1) ϵ is increasing.

(2) $\lim_{r\to 0} \epsilon(r) = 0$.

(3) $\left\|\frac{1}{2}(x+y)\right\| \le 1 - \epsilon(\|x-y\|)$ for all $x,y \in B_1(0)$, the unit ball of X.

Theorem 5.4 (Clarkson 1936). *Let X be a uniformly convex Banach space, K a closed convex subset of X, and z any point of X. Then the minimization problem (?) has a unique solution $y_0 \in K$.*

Proof. If $z \in K$ then $y_0 = z$ is the solution, and is clearly unique. When $z \notin K$, we may assume $z = 0$ (translating z and K if necessary). Let

$$s = \inf_{y\in K} \|y\|.$$

So $s > 0$. Now let $y_n \in K$ be a minimizing sequence, so

$$\|y_n\| \to s.$$

Now let $x_n = y_n/\|y_n\|$, and consider

$$\frac{1}{2}(x_n + x_m) = \frac{1}{2\|y_n\|}y_n + \frac{1}{2\|y_m\|}y_m = \left(\frac{1}{2\|y_n\|} + \frac{1}{2\|y_m\|}\right)(ty_n + (1-t)y_m)$$

for suitable t. So $ty_n + (1-t)y_m \in K$ and

$$\|ty_n + (1-t)y_m\| \ge s.$$

Thus

$$1 - \epsilon(\|x_n - x_m\|) \ge \frac{1}{2}\left(\frac{s}{\|y_n\|} + \frac{s}{\|y_m\|}\right) \to 1 \qquad .$$

We conclude that x_n is a Cauchy sequence, from which it follows that y_n is Cauchy. The limit $y_0 \in \lim_n y_n$ exists in K since X is complete and K is closed. Clearly $\|y_0\| = s$.

Warning: Not every Banach space is uniformly convex. For example, the space $C(D)$ of continuous functions on a compact set D is not uniformly convex. It may even happen that

$$\left\|\frac{1}{2}(f+g)\right\|_\infty = 1$$

for unit vectors f and g. (They need only have disjoint support.) Lax gives an example of a closed convex set in $C[-1,1]$ in which the minimization problem (?) has no solution.

It can also happen that a solution exists but is not unique. For example, in $C[-1,1]$ let K = {functions that vanish on $[-1,0]$}. and let $f = 1$ on $[-1,1]$. Clearly

$$\sup_x |f(x) - g(x)| \ge 1 \quad \forall g \in K,$$

and the distance 1 is attained for any $g \in K$ that satisfies $0 \le g(x) \le 1$.

Linear Functionals on a Banach Space

Definition 6.1. A linear functional $\ell : X \to F$ on a normed space X over $F = \mathbb{R}$ or $\mathbb{C}$ is <u>bounded</u> if there is $c < \infty$ such that

$$|\ell(x)| \le c\|x\| \qquad \forall x \in X.$$

The inf over all such c is the <u>norm</u> $\|\ell\|$ of ℓ,

$$\|\ell\| = \sup_{x \neq 0,\ x \in X} \frac{|\ell(x)|}{\|x\|}. \tag{?}$$

Theorem 6.1. *A linear functional ℓ on a normed space is bounded if and only if it is continuous.*

Proof. It is useful to recall that

Theorem 6.2. *Let X,Y be metric spaces. Then $f : X \to Y$ is continuous if and only if $f(x_n)$ is a convergent sequence in Y whenever x_n is convergent in X.*

Remark. Continuity $\Longrightarrow$ the sequence condition in any topological space. That the sequence condition $\Longrightarrow$ continuity follows from the fact that the topology has a countable basis at a point. (In a metric space, $B_{2^{-n}}(x)$, say.) Specifically, suppose the function is not continuous. Then there is an open set $U \subset Y$ such that $f^{-1}(U)$ is not open. So there is $x \in f^{-1}(U)$ such that for all n $B_{2^{-n}}(x) \not\subset f^{-1}(U)$. Now let x_n be a sequence such that

(1) $x_n \in B_{2^{-n}}(x) \setminus f^{-1}(U)$ for n odd.
(2) $x_n = x$ for n even.

Clearly $x_n \to x$. However, $f(x_n)$ cannot converge since $\lim_k f(x_{2k}) = f(x)$ and any limit point of $f(x_{2k+1})$ lies in the closed set U^c containing all the points $f(x_{2k+1})$.

In the normed space context, note that

$$|\ell(x_n) - \ell(x)| \le \|\ell\|\|x_n - x\|,$$

so boundedness certainly implies continuity.

Conversely, if ℓ is unbounded then we can find vectors x_n so that $\ell(x_n) \ge \sqrt{n}\|x_n\|$. Since this inequality is homogeneous under scaling, we may suppose that $\|x_n\| = 1/\sqrt{n}$, say. Thus $x_n \to 0$ and $\ell(x_n) \to \infty$, so ℓ is not continuous.

The set X' of all bounded linear functionals on X is called the <u>dual of X</u>. It is a linear space, and in fact a normed space under the norm (?). (It is straightforward to show that (?) defines a norm.)

Theorem 6.3. *The dual X' of a normed space X is a Banach space.*

Proof. We need to show X' is complete. Suppose ℓ_m is a Cauchy Sequence. Then for each $x \in X$ we have

$$|\ell_n(x) - \ell_m(x)| \le \|\ell_n - \ell_m\|\|x\|,$$

so $\ell_n(x)$ is a Cauchy sequence of scalars. Let $\ell(x) = \lim_{n\to\infty} \ell_n(x)$ $\forall x \in X$.

It is easy to see that ℓ is linear. Let us show that it is bounded. Since $|\|\ell_n\| - \|\ell_m\|| \le \|\ell_n - \ell_m\|$ (this follows from sub-additivity), we see that the sequence $\|\ell_n\|$ is Cauchy, and thus bounded. So,

$$|\ell(x)| \le \sup_n \|\ell_n\| \, \|x\|$$

and ℓ is bounded. Similarly,

$$|\ell_n(x) - \ell(x)| \le \sup_{m\ge n} \|\ell_n - \ell_m\|\|x\|,$$

so

$$\|\ell_n - \ell\| \le \sup_{m\ge n} \|\ell_n - \ell_m\|$$

and it follows that $\ell_n \to \ell$.

Of course, all of this could be vacuous. How do we know that there are any bounded linear functionals? Here the Hahn-Banach theorem provides the answer.

Theorem 6.4. *Let $y_1,...,y_N$ be N linearly independent vectors in a normed space X and $\alpha_1,...,\alpha_N$ arbitrary scalars. Then there is a bounded linear functional $\ell \in X'$ such that $\ell(y_j) = \alpha_j$, $j = 1,...,N$.*

Proof. Let $Y = \operatorname{span}\{y_1,...,y_N\}$ and define ℓ on Y by

$$\ell\Big(\sum_{j=1}^{N} b_j\Big) = \sum_{j=1}^{N} b_j \alpha_j.$$

(We use linear independence here to guarantee that ℓ is well defined.) Clearly, ℓ is linear. Furthermore, since Y is finite dimensional ℓ is bounded.

(Explicitly, since any two norms on a finite dimensional space are equivalent, we can find c such that

$$\sum_j |b_j| \, \|y_j\| \le c \left\| \sum_j b_j y_j \right\| .$$

Thus, $\ell(y) \le c \sup_j |\alpha_j| \, \|y\|$ for $y \in Y$.)

Thus ℓ is a linear functional on Y dominated by the norm $\|\cdot\|$. By the Hahn-Banach theorem, it has an extension to X that is also dominated by $\|\cdot\|$, i.e., that is bounded.

A closed subspace Y of a normed space X is itself a normed space. If X is a Banach space, so is Y. A linear functional $\ell \in X'$ on X can be restricted to Y and is still bounded. That is there is a restriction map $R : X' \to Y'$ such that

$$R(\ell)(y) = \ell(y) \qquad \forall y \in Y.$$

It is clear that R is a linear map and that

$$\mathrm{k}R(\grave{})\mathrm{k} \leq \mathrm{k}\grave{}\mathrm{k}.$$

The Hahn-Banach Theorem shows that R is surjective. On the other hand, unless $Y = X$, the kernel of R is certainly non-trivial. To see this, let x be a vector in $X \setminus Y$ and define ` on span$\{x\}$ ∪ Y by

$$\grave{}(ax + y) = a \qquad \forall a \in F \text{ and } y \in Y.$$

Then ` is bounded on the closed subspace span$\{x\}$ ∪ Y and by Hahn-Banach has a closed extension. Clearly $R(\grave{}) = 0$. The kernel of R is denoted $Y^{\perp}$, so

$$Y^{\perp} = \{\grave{} \in X^0 : \grave{}(y) = 0 \qquad \forall y \in Y\},$$

and is a Banach space.

Now the quotient space X/Y is defined to be the set of "cosets of Y,"

$$\{Y + x : x \in X\}.$$

The coset $Y + x$ is denoted $[x]$. The choice of label x is, of course, not unique as $x + y$ with $y \in Y$ would do just as well. It is a standard fact that

$$[x_1] + [x_2] = [x_1 + x_2]$$

gives a well defined addition on Y/X so that it is a linear space.

Lemma 6.5. *If Y is a closed subspace of a normed space then*

$$\mathrm{k}[x]\mathrm{k} = \inf_{y \in Y} \mathrm{k}x + y\mathrm{k}$$

is a norm on X/Y . If X is a Banach space, so is X/Y .

Proof. Exercise.

A bounded linear functional that vanishes on Y , that is an element $\grave{} \in Y^{\perp}$, can be understood as a linear functional on X/Y since the definition

$$\grave{}([x]) = \grave{}(x)$$

is unambiguous. Thus we have a map $J : Y^{\perp} \to (X/Y)^0$ defined by $J(\grave{})([x]) = \grave{}(x)$. Conversely, there is a bounded linear map $\Pi : X \to X/Y$ given by $\Pi(x) = [x]$ and any linear functional $\grave{} \in (X/Y)^0$ pulls back to a bounded linear functional $\grave{} \circ \Pi$ in $Y^{\perp}$. Clearly $J(\grave{} \circ \Pi) = \grave{}$. Thus we have, loosely, that

$$(X/Y)^0 = Y^{\perp}.$$

Isometries of a Banach Space

Definition 7.1. Let X,Y be normed spaces. A linear map $T : X \to Y$ is <u>bounded</u> if there is $c > 0$ such that $\mathrm{k}T(x)\mathrm{k} \leq c\mathrm{k}x\mathrm{k}$. The <u>norm of T</u> is the smallest such c, that is

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.$$

Theorem 7.1. *A linear map $T : X \to Y$ between normed spaces X and Y is continuous if and only if it is bounded.*

Remark. The proof is a simple extension of the corresponding result for linear functionals.

An isometry of normed spaces X and Y is a map $M : X \to Y$ such that

(1) M is surjective.
(2) k$M(x) - M(y)$k = k$x - y$k.

Clearly translations $T_u : X \to X$, $T_u(x) = x + u$ are isometries of a normed linear space. A linear map $T : X \to Y$ is an isometry if T is surjective and

$$\mathrm{k}T(x)\mathrm{k} = \mathrm{k}x\mathrm{k} \qquad \forall x \in X.$$

A map $M : X \to Y$ is <u>affine</u> if $M(x) - M(0)$ is linear. So, M is affine if it is the composition of a linear map and a translation.

Theorem 7.2 (Mazur and Ulam 1932). *Let X and Y be normed spaces over* R*. Any isometry $M : X \to Y$ is an affine map.*

Remark. The theorem conclusion does <u>not</u> hold for normed spaces over C. In that context any isometry is a real -affine map ($M(x) - M(0)$ is real linear), but not necessarily

a complex-affine map. For example on $C([0,1],\mathrm{C})$ the map f 7→ $\bar{f}$ (complex conjugation) is an isometry and is not complex linear.

Proof. It suffices to show $M(0) = 0 \implies M$ is linear. To prove linearity it suffices to show

$$M\left(\frac{1}{2}(x+y)\right) = \frac{1}{2}(M(x) + M(y)) \quad \forall x, y \in X.$$

(Why?)

Let x and y be points in X and $z = \frac{1}{2}(x+y)$. Note that

$$\|x - z\| = \|y - z\| = \frac{1}{2}\|x - y\|,$$

so z is "half-way between x and y." Let

$$x^0 = M(x),\ y^0 = M(y),\ z^0 = M(z). \text{ We need}$$

to show $2z^0 = x^0 + y^0$. Since M is an isometry,

$$\|x' - z'\| = \|y' - z'\| = \frac{1}{2}\|x' - y'\|,$$

and all of these are equal to $\frac{1}{2}\|x - y\|$. So z^0 is "half-way between x^0 and y^0." It may happen that $\frac{1}{2}(x' + y')$ is the unique point of Y with this property (in which case we are done). this happens, for instance, if the norm in Y is <u>strictly sub-additive</u>, meaning

$$\beta x^0\ 6\!= \alpha y^0 \implies \mathrm{k}x^0 + y^0\mathrm{k} < \mathrm{k}x^0\mathrm{k} + \mathrm{k}y^0\mathrm{k}.$$

In general, however, there may be a number of points "half-way between x^0 and y^0." So, let

$$A_1 = \{u \in X \;:\; \|x-u\| = \|y-u\| = \frac{1}{2}\|x-y\|\},$$

and

$$A'_1 = \{u' \in Y \;:\; \|x'-u'\| = \|y'-u'\| = \frac{1}{2}\|x'-y'\|\}$$.

Since M is an isometry, we have $A'_1 = M(A_1)$. Let d_1 denote the diameter of A_1,

$$d_1 = \sup_{u,v \in A_1} \mathrm{k}u - v\mathrm{k}.$$

This is also the diameter of A'_1. Now, let

$$A_2 = \left\{u \in A_1 \;:\; v \in A_1 \implies \|u-v\| \le \frac{1}{2}d_1\right\},$$

the set of "centers of A_1." Note that $z \in A_2$ since if $u \in A_1$ then $2z - u \in A_1$:

$$\mathrm{k}x - (2z-u)\mathrm{k} = \mathrm{k}u - y\mathrm{k} = \mathrm{k}x - u\mathrm{k} = \mathrm{k}y - (2z-u)\mathrm{k}.$$

Similarly, let

$$A'_2 = \left\{u' \in A'_1 \;:\; v' \in A'_1 \implies \|u'-v'\| \le \frac{1}{2}d_1\right\}.$$

Again, since M is an isometry we have $A'_2 = M(A_2)$. In a similar way, define decreasing sequences of sets, A_j and $A^0{}_j$, inductively by

$$A_j = \{u \in A_{j-1} \;:\; v \in A_{j-1} \implies \|u-v\| \le \frac{1}{2}\operatorname{diam}(A_{j-1})\},$$

and

$$A'_j = \{u' \in A'_{j-1} \;: v' \in A'_{j-1} \implies \|u'-v'\| \le \frac{1}{2}\operatorname{diam}(A'_{j-1})\}.$$

Again $M(A_j) = A^0{}_j$ and $z \in A_j$ since A_{j-1} is invariant under inversion around z: $u \in A_{j-1} \Longrightarrow 2z - u \in A_{j-1}$. Since $\operatorname{diam}(A_j) \le 2^{1-j}d_1$ we conclude that

$$\backslash_{j=1}^{\infty} = \{z\}, \text{ and. } \qquad \bigcap_{j=1}^{\infty} A'_j = \left\{\frac{1}{2}(x'+y')\right\}$$

Since $z^0 \in A^0{}_j$ for all j, (?) follows.